Journal of Statistical Physics, Vol. 106, Nos. 3/4, February 2002 (© 2002)

The Two Dimensional Hubbard Model at Half-Filling. I. Convergent Contributions

V. Rivasseau[1]

Received July 9, 2001; accepted October 4, 2001

We prove analyticity theorems in the coupling constant for the Hubbard model at half-filling. The model in a single renormalization group slice of index i is proved to be analytic in λ for $|\lambda| \leqslant c/i$ for some constant c, and the skeleton part of the model at temperature T (the sum of all graphs without two point insertions) is proved to be analytic in λ for $|\lambda| \leqslant c/|\log T|^2$. These theorems are necessary steps towards proving that the Hubbard model at half-filling is *not* a Fermi liquid (in the mathematically precise sense of Salmhofer).

KEY WORDS: Superconductivity; Fermi liquid; Luttinger liquid; Hubbard model.

1. INTRODUCTION

Constructive renormalization group approach to the Fermi systems of condensed matter[(1–3)] is an ongoing program to study quite systematically the properties of interacting non-relativistic Fermions at finite density in one, two or three dimensions. In one dimension interacting Fermions have been proved to form a Luttinger liquid until zero temperature.[(4, 5)] The simplest interacting two-dimensional model for Fermions, namely the jellium model, has been recently shown to be a Fermi liquid[(6, 7)] above the critical temperature, in the sense of Salmhofer's criterion.[(8)] The next most natural model in two dimensions is the Hubbard model on a square lattice at half-filling considered in this paper. This model presents the interesting features of a square Fermi surface with nesting vectors and van Hove singularities. It has also particle-hole symmetry, which preserves the Fermi surface under the renormalization group flow. For all these reasons it is the

[1] Centre de Physique Théorique, CNRS UMR 7644, École Polytechnique, 91128 Palaiseau Cedex, France; e-mail: Vincent.Rivasseau@cpht.polytechnique

best candidate for a first example in two dimensions of a Fermionic system which is *not* a Fermi liquid, but rather some kind of Luttinger liquid with logarithmic corrections.

To study the Fermi versus Luttinger behavior according to Salmhofer's criterion, one must prove analyticity in the coupling constant in a domain above some critical temperature, and study whether the first and second derivatives of the self-energy in Fourier space are uniformly bounded or not in that analyticity domain.[2] This analysis may be conveniently decomposed into four main steps of increasing difficulty:

(A) control of the model in a single slice;

(B) control of the model without divergent subgraphs in many slices;

(C) control of the two point function renormalization;

(D) study of the first and second derivatives of the self-energy.

This program is completed only at the moment for the two-dimensional jellium model: steps A and B were performed in ref. 12 and steps C and D in ref. 7. For the three dimensional jellium model, steps A and B have been successively completed in refs. 13 and 14. For the half-filled Hubbard model we perform in this paper steps A and B. We use an angular decomposition of the model into "sectors" that are very different from the jellium case. We write the momentum conservation rules in terms of these sectors. Then we prove that the sum over all graphs with momenta restricted to the ith slice of the renormalization group is analytic for $|\lambda| \leqslant const/i$ (step A). We prove two theorems corresponding to step B. The first one states that the completely convergent part of the theory, namely the sum over all graphs which do not contain two-particle and four particle subgraphs with external legs closer to the singularity than their internal legs, is analytic for $|\lambda \log T| \leqslant const$. The second result states that the "biped-free" part of the theory, namely the sum over all graphs which do not contain two-particle subgraphs with external legs closer to the singularity than their internal legs, is analytic for $|\lambda \log^2 T| \leqslant const$. We remark that this last domain of analyticity is the expected optimal domain for the full theory. We remark also that these domains are smaller than the domains for the jellium case which are respectively $|\lambda| \leqslant const$ for the single slice or completely convergent

[2] In two or more dimensions perturbation theory can generically work only above some critical temperature, so the Fermi liquid behavior cannot persist until zero temperature, except for very particular models with a Fermi surface which is not parity invariant. There is an ongoing program to study these models in two dimensions.[9–11]

theory, and $|\lambda \log T| \leqslant const$ for the biped free part of the theory or the full theory with an appropriate mass-counterterm.

Finally we remark also that since we expect the half-filled Hubbard model *not* to be a Fermi liquid, step D in that case should consist of a proof that the second momentum derivative of the self-energy is *not* uniformly bounded in that domain of analyticity. This requires a lower bound showing the divergence of this quantity near the corner $\lambda, T \to 0$ rather than an upper bound.

For a very simple introduction to constructive Fermionic theory, we recommend ref. 15. We will also use without too much further explanations the Taylor tree formulas that are developed in detail in ref. 16. It would also be useful if the reader has already some familiarity with the basics of multiscale expansions[17] and with constructive Fermionic renormalization, as, e.g., developed in ref. 18; but we will try to remain as simple and self-contained as possible.

2. MODEL AND NOTATIONS

A finite temperature Fermionic model has a propagator $C(x, \bar{x})$ where $x = (x_0, \vec{x})$, which is translation invariant. By some slight abuse of notations we may therefore write it either $C(x - \bar{x})$ or $C(x, \bar{x})$, where the first point corresponds to the field and the second one to the antifield. This propagator at finite temperature is antiperiodic in the variable x_0 with antiperiod $\frac{1}{T}$, hence its Fourier transform depends on discrete values (called the Matsubara frequencies):

$$k_0 = \frac{2n+1}{\beta} \pi, \qquad n \in \mathbb{Z}, \tag{2.1}$$

where $\beta = 1/T$ (we take $\hbar = k = 1$). Remark that only odd frequencies appear, because of antiperiodicity.

The Hubbard model lives on the square lattice $\mathbb{Z}^2$, so that the three dimensional vector $x = (x_0, \vec{x})$ is such that $\vec{x} = (n_1, n_2) \in \mathbb{Z}^2$. From now on we write v_1 and v_2 for the two components of a vector $\vec{v}$ along the two axis of the lattice.

At half-filling and finite temperature T, the Fourier transform of the propagator of the Hubbard model is:

$$\hat{C}_{ab}(k) = \delta_{ab} \frac{1}{ik_0 - e(\vec{k})}, \qquad e(\vec{k}) = \cos k_1 + \cos k_2, \tag{2.2}$$

where $a, b \in \{\uparrow, \downarrow\}$ are the spin indices. The vector $\vec{k}$ lives on the two-dimensional torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. Hence the real space propagator is

$$C_{ab}(x) = \frac{1}{(2\pi)^2 \beta} \sum_{k_0} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \, e^{ikx} \hat{C}_{ab}(k). \tag{2.3}$$

The notation $\sum_{k_0}$ means really the discrete sum over the integer n in (2.1). When $T \to 0$ (which means $\beta \to \infty$) k_0 becomes a continuous variable, the corresponding discrete sum becomes an integral, and the corresponding propagator $C_0(x)$ becomes singular on the Fermi surface defined by $k_0 = 0$ and $e(\vec{k}) = 0$. This Fermi surface is a square of side size $\sqrt{2}\,\pi$ (in the first Brillouin zone) joining the corners $(\pm\pi, 0), (0, \pm\pi)$. We call this square the Fermi square, its corners and faces are called the Fermi faces and corners. Considering the periodic boundary conditions, there are really four Fermi faces, but only two Fermi corners.

In the following to simplify notations we will write:

$$\int d^3k \equiv \frac{1}{\beta} \sum_{k_0} \int d^2k, \qquad \int d^3x \equiv \frac{1}{2} \int_{-\beta}^{\beta} dx_0 \sum_{\vec{x} \in \mathbb{Z}^2} . \tag{2.4}$$

In determining the spatial decay we recall that by antiperiodicity

$$C(x) = f(x_0, \vec{x}) := \sum_{m \in \mathbb{Z}} (-1)^m \, C_0 \left(x_0 + \frac{m}{T}, \vec{x} \right). \tag{2.5}$$

where C_0 is the propagator at $T = 0$. Indeed the function f is antiperiodic and its Fourier transform is the right one.

The interaction of the Hubbard model is simply

$$S_V = \lambda \int_V d^3x \left(\sum_a \bar{\psi}_a \psi_a \right)^2, \tag{2.6}$$

where $V := [-\beta, \beta] \times V'$ and V' is an auxiliary finite volume cutoff in two dimensional space that will be sent later to infinity. Remark that in (2.1) $|k_0| \geqslant \pi/\beta \neq 0$ hence the denominator in $C(k)$ can never be 0 at non zero temperature. This is why the temperature provides a natural infrared cut-off.

2.1. Scale Analysis

The theory has a natural lattice spatial cutoff. To implement the renormalization group analysis, we introduce as usually a compact support function $u(r) \in \mathscr{C}_0^\infty(\mathrm{R})$ (it is convenient to choose it to be Gevrey[19] of order $\alpha < 1$ so as to ensure fractional exponential decrease in the dual space) which satisfies:

$$u(r) = 0 \quad \text{for } |r| > 2; \qquad u(r) = 1 \quad \text{for } |r| < 1. \tag{2.7}$$

With this function, given a constant $M \geqslant 2$, we can construct a partition of unity

$$\begin{aligned} 1 &= \sum_{i=0}^{\infty} u_i(r) \qquad \forall r \neq 0; \\ u_0(r) &= 1 - u(r); \qquad u_i(r) = u(M^{2(i-1)}r) - u(M^{2i}r) \quad \text{for} \quad i \geqslant 1. \end{aligned} \tag{2.8}$$

The propagator is then divided into slices according to this partition

$$C(k) = \sum_{i=0}^{\infty} C_i(k) \tag{2.9}$$

where

$$C_i(k) = C(k)\, u_i[k_0^2 + e^2(\vec{k})]. \tag{2.10}$$

(indeed $k_0^2 + e^2(\vec{k}) \geqslant T^2 > 0$).

In a slice of index i the cutoffs ensure that the size of $k_0^2 + e^2(\vec{k})$ is roughly M^{-2i}. More precisely in the slice i we must have

$$M^{-2i} \leqslant k_0^2 + e^2(\vec{k}) \leqslant 2M^2 M^{-2i}. \tag{2.11}$$

The corresponding domain is a three dimensional volume whose section through the $k_0 = 0$ plane is the shaded region pictured in Fig. 1.

Remark that at finite temperature, the propagator C_i vanishes for $i \geqslant i_{\max}(T)$ where $M^{i_{\max}(T)} \simeq 1/T$ (more precisely $i_{\max}(T) = E(\log \frac{M\sqrt{2}}{\pi T} / \log M)$, where E is the integer part), so there is only a finite number of steps in the renormalization group analysis.

Let us state first our simplest result, for a theory whose propagator is only C_i, hence corresponds to a generic step of the renormalization group:[3]

[3] In the following we assume $i \geqslant 1$. Indeed the first slice $i = 0$ is somewhat peculiar because of the unboundedness of the Matsubara frequencies, which requires a little additional care.

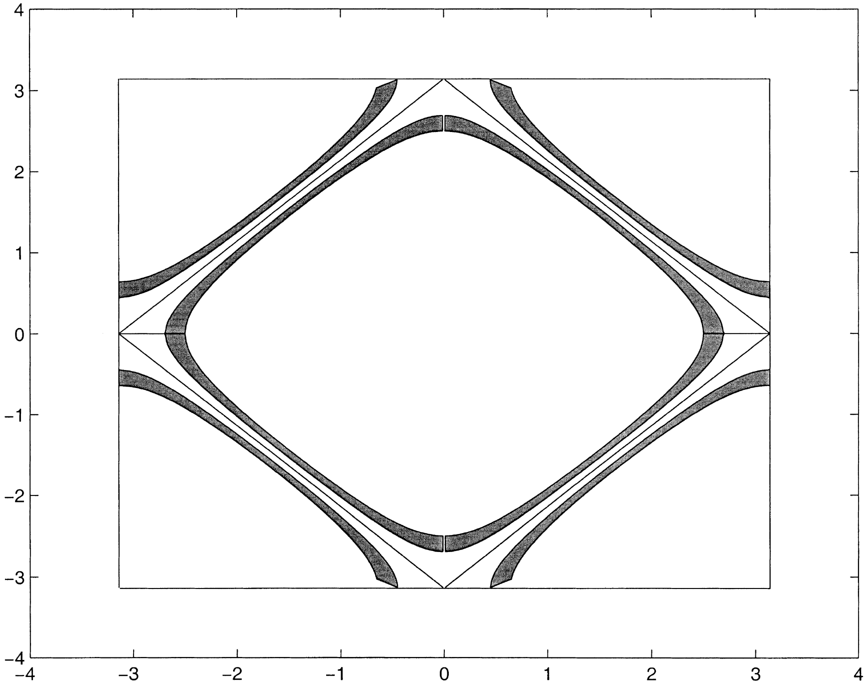

Fig. 1. A single slice of the renormalization group.

Theorem 1. The Schwinger functions of the theory with propagator C_i and interaction (2.6) are analytic in λ in a disk of radius R_i which is at least c/i for a suitable constant c:

$$R_i \geqslant c/i. \tag{2.12}$$

The rest of this section is devoted to definitions and preliminary lemmas about sectors, their scaled decay and momentum conservation rules. Although Theorem 1 applies to a single slice, its proof nevertheless requires some kind of multiscale analysis, which is done in Section 3. Our next results, Theorems 2 and 3, which bound the sum over all "convergent contributions," that is without divergent two point insertions, are slightly more technical to state, but their proof is almost identical to that of Theorem 1. They are postponed to Section 4.

As discussed in the introduction this result is a first step towards the full analysis of the model in the regime $|\lambda \log^2 T| \leqslant const.$, and a rigorous proof that it is not a Fermi liquid in the sense of Salmhofer.

2.2. Sectors

The "angular" analysis is completely different from the jellium case. We remark first that in our slice, $k_0^2 + e^2(\vec{k})$ is of order M^{-2i}, but this does not fix the size of $e^2(\vec{k})$ itself, which can be of order M^{-2j} for some $j \geqslant i$. In order for sectors defined in momentum space to correspond to propagators with dual decay in direct space, it is essential that their length in the tangential direction is not too big, otherwise the curvature is too strong for stationary phase methods to apply. This was discussed first in ref. 12. This leads us to study the curve $(\cos k_1 + \cos k_2)^2 = M^{-2j}$ for arbitrary $j \geqslant i$. We can by symmetry restrict ourselves to the region $0 \leqslant k_1 \leqslant \pi/2$, $k_2 > 0$. It is then easy to compute the curvature radius of that curve, which is

$$R = \frac{(\sin^2 k_1 + \sin^2 k_2)^{3/2}}{|\cos k_1 \sin^2 k_2 + \cos k_2 \sin^2 k_1|}. \tag{2.13}$$

We can also compute the distance $d(k_1)$ to the critical curve $\cos k_1 + \cos k_2 = 0$, and the width $w(k_1)$ of the band $M^{-j} \leqslant |\cos k_1 + \cos k_2| \leqslant \sqrt{2}\, M.M^{-j}$. We can then easily check that

$$d(k_1) \simeq w(k_1) \simeq \frac{M^{-j}}{M^{-j/2} + k_1}, \tag{2.14}$$

$$R(k_1) \simeq \frac{k_1^3 + M^{-3j/2}}{M^{-j}}, \tag{2.15}$$

where $f \simeq g$ means that on the range $0 \leqslant k_1 \leqslant \pi/2$ we have inequalities $cf \leqslant g \leqslant df$ for some constants c and d.

Defining the anisotropic length

$$l(k_1) = \sqrt{w(k_1)\, R(k_1)} \simeq M^{-j/2} + k_1, \tag{2.16}$$

the condition which generalizes the one used in ref. 12 is that the sector length should not be bigger than that anisotropic length. (The analysis in ref. 12 was of course simpler since for a circular Fermi surface the curvature radius is constant). This leads to the idea that k_1 or an equivalent quantity should be sliced according to a geometric progression from 1 to $M^{-j/2}$ to form the angular sectors in this model.

For symmetry reasons it is convenient to introduce a new orthogonal but not normal basis in momentum space (e_+, e_-), defined by $e_+ = (1/2)(\pi, \pi)$ and $e_- = (1/2)(-\pi, \pi)$. Indeed if we call (k_+, k_-) the coordinates of a momentum k in this basis, the Fermi surface is given by the simple equations $k_+ = \pm 1$ or $k_- = \pm 1$. This immediately follows from the identity

$$\cos k_1 + \cos k_2 = 2 \cos(\pi k_+/2) \cos(\pi k_-/2). \tag{2.17}$$

(Note however that the periodic b.c. are more complicated in that new basis). Instead of slicing $e(\vec{k})$ and k_1, it is then more symmetric to slice directly $\cos(\pi k_+/2)$ and $\cos(\pi k_-/2)$.

Guided by these considerations we introduce the partition of unity

$$1 = \sum_{s=0}^{i} v_s(r); \qquad \begin{cases} v_0(r) = 1 - u(M^2 r) \\ v_s = u_{s+1} & \text{for} \quad 1 \leqslant s \leqslant i-1 \\ v_i(r) = u(M^{2i} r) \end{cases} \tag{2.18}$$

and define

$$C_i(k) = \sum_{\sigma = (s_+, s_-)} C_{i,\sigma}(k) \tag{2.19}$$

where

$$C_{i,\sigma}(k) = C_i(k)\, v_{s_+}[\cos^2(\pi k_+/2)]\, v_{s_-}[\cos^2 \pi k_-/2)]. \tag{2.20}$$

We remark that using (2.11) in order for $C_{i,\sigma}$ not to be 0, we need to have $s_+ + s_- \geqslant i-2$. We define the "depth" $l(\sigma)$ of a sector to be $l = s_+ + s_- - i + 2$.

To get a better intuitive picture of the sectors, we remark that they can be classified into different categories:

— the sectors $(0, i)$ and $(i, 0)$ are called the middle-face sectors

— the sectors (s, i) and (i, s) with $0 < s < i$ are called the face sectors

— the sector (i, i) is called the corner sector

— the sectors (s, s) with $(i-2)/2 \leqslant s < i$ are called the diagonal sectors

— the others are the general sectors

Finally the general or diagonal sectors of depth 0 for which $s_+ + s_- = i-2$ are called border sectors.

If we consider the projection onto the (k_+, k_-) plane, taking into account the periodic b.c. of the Brillouin zone, the general and diagonal sectors have 8 connected components, the face sectors have 4 connected components, the middle face sectors and the corner sector have 2 connected components. In the three dimensional space-time, if we neglect the discretization of the Matsubara frequencies, these numbers would double except for the border sectors.

2.3. Scaled Decay

Lemma 1. Using Gevrey cutoffs of degree $\alpha < 1$, the propagator $C_{i,\sigma}$ obeys the scaled decay

$$|C_{i,\sigma}| \leqslant c.M^{-i-l} e^{-c[d_{i,\sigma}(x,y)]^{\alpha}} \tag{2.21}$$

where

$$d_{i,\sigma}(x,y) = \{M^{-i}\,|x_0 - y_0| + M^{-s_+}\,|x_+ - y_+| + M^{-s_-}\,|x_- - y_-|\}, \tag{2.22}$$

and c is some constant.

Proof. This is essentially Fourier analysis and integration by parts. If $x = (n_1, n_2) \in \mathbb{Z}^2$, we define $(x_+, x_-) = (\pi/2)(n_1 + n_2, n_2 - n_1)$. The vector (x_+, x_-) then belongs to $(\pi/2)\,\mathbb{Z}^2$ but with the additional condition that x_+ and x_- have the same parity.

Defining, for $X \in [(\pi/2)\,\mathbb{Z}]^2$

$$\begin{aligned} D_{i,\sigma}(X) = {} & (1/2)\frac{1}{8\beta}\sum_{k_0}\int_{-2}^{+2} dk_+ \int_{-2}^{+2} dk_-\; e^{i(k_0 x_0 + k_+ x_+ + k_- x_-)} \\ & \times \frac{u_i[k_0^2 + 4\cos^2(\pi k_+/2)\cos^2(\pi k_-/2)]}{ik_0 - 2\cos(\pi k_+/2)\cos(\pi k_-/2)} \\ & \times v_{s_+}[\cos^2(\pi k_+/2)]\; v_{s_-}[\cos^2(\pi k_-/2)] \end{aligned} \tag{2.23}$$

we note that $C_{i,\sigma}(X) = D_{i,\sigma}(X)$ for X satisfying the parity condition.

(Remember the Jacobian $\frac{\pi^2}{2}$ from $dk_1\,dk_2$ to $dk_+\,dk_-$, and the initial domain of integration that is doubled.)

The volume of integration trivially gives a factor M^{-i} for the k_0 sum and factors M^{-s_+} and M^{-s_-} for the k_+ and k_- integration (see (2.28) later). The integrand is trivially bounded by M^i on the integration domain, and this explains the prefactor cM^{-i-l} in (2.21).

We then apply standard integration by parts techniques to formulate the decay. From, e.g., Lemma 10 in ref. 6 we know that to obtain the scaled decay of Lemma 1 we have only to check the usual derivative bounds in Fourier space:

$$\left\| \frac{\partial^{n_0}}{\partial k_0^{n_0}} \frac{\partial^{n_+}}{\partial k_+^{n_+}} \frac{\partial^{n_-}}{\partial k_0^{n_-}} \hat{D}_{i,\sigma} \right\| \leqslant A.B^n M^{in_0} M^{s_+ n_+} M^{s_- n_-} (n!)^{1/\alpha} \tag{2.24}$$

where $n = n_0 + n_+ + n_-$, and the derivative $\frac{\partial}{\partial k_0}$ really means the natural finite difference operator $(1/2\pi T)(f(k_0+2\pi T) - f(k_0))$ acting on the discrete Matsubara frequencies. The norm is the ordinary sup norm.

But from (2.23),

$$\hat{D}_{i,\sigma}(k) = \frac{1}{16\beta} \frac{u_i[k_0^2 + 4\cos^2(\pi k_+/2)\cos^2(\pi k_-/2)]}{ik_0 - 2\cos(\pi k_+/2)\cos(\pi k_-/2)}$$
$$\times v_{s_+}[\cos^2(\pi k_+/2)]\, v_{s_-}[\cos^2(\pi k_-/2)] \tag{2.25}$$

and the derivatives are bounded easily using the standard rules for derivation, product and composition of Gevrey functions, or by hand, using the support properties of the v_{s_+} and v_{s_-} functions. For instance a derivative $\frac{\partial}{\partial k_+}$ can act on the $v_{s_+}[\cos^2(\pi k_+/2)]$ factor, in which case it is easily directly bounded by cM^{s_+} for some constant c. When it acts on $u_i[k_0^2 + 4\cos^2(\pi k_+/2)\cos^2(\pi k_-/2)]$ it is easily bounded by $c.M^{2i - s_+ - 2s_-}$ hence by $c.M^{s_+}$, using the relation $s_+ + s_- \geqslant i-2$. When it acts on the denominator $[ik_0 - 2\cos(\pi k_+/2)\cos(\pi k_-/2)]^{-1}$. it is bounded by $c.M^{i-s_-}$, hence again by $c.M^{s_+}$, using the relation $s_+ + s_- \geqslant i-2$. Finally when it acts on a $\cos(\pi k_+/2)$ created by previous derivations, it costs directly $c.M^{s_+}$. The factorial factor $(n!)^{1/\alpha}$ in (2.24) comes naturally from deriving the cutoffs, which are Gevrey functions of order α; deriving other factors give smaller factorials (with power 1 instead of $1/\alpha$).

Finally a last remark: to obtain the lemma for the last slice, $i = i_{\max}(T)$, one has to take into account the fact that x_0 lies in a compact circle, so that there is really no long-distance decay to prove.

2.4. Support Properties

If $C_{i,\sigma}(k) \neq 0$, the momentum k must obey the following bounds:

$$|k_0| \leqslant \sqrt{2}\, M M^{-i} \tag{2.26}$$

$$\begin{cases} M^{-1} \leqslant |\cos(\pi k_\pm/2)| \leqslant 1 & \text{for } s_\pm = 0, \\ M^{-s_\pm - 1} \leqslant |\cos(\pi k_\pm/2)| \leqslant \sqrt{2}\, M^{-s_\pm} & \text{for } 1 \leqslant s_\pm \leqslant i-1, \\ |\cos(\pi k_\pm/2)| \leqslant \sqrt{2}\, M^{-i} & \text{for } s_\pm = i. \end{cases} \tag{2.27}$$

In the support of our slice in the first Brillouin zone we have $|k_+| < 2$ and $|k_-| < 2$ (this is not essential but the inequalities are strict because $i \geqslant 1$). It is convenient to associate to any such component $k_\pm$ a kind of "fractional part" called $q_\pm$ defined by $q_\pm = k_\pm - 1$ if $k_\pm \geqslant 0$ and $q_\pm = k_\pm + 1$ if $k_\pm < 0$, so that $0 \leqslant |q_\pm| \leqslant 1$. Then the bounds translate into

$$\begin{cases} 2/\pi M \leqslant |q_\pm| \leqslant 1 & \text{for} \quad s_\pm = 0, \\ 2M^{-s_\pm}/\pi M \leqslant |q_\pm| \leqslant \sqrt{2}\, M^{-s_\pm} & \text{for} \quad 1 \leqslant s_\pm \leqslant i-1, \\ |q_\pm| \leqslant \sqrt{2}\, M^{-i} & \text{for} \quad s_\pm = i. \end{cases} \tag{2.28}$$

2.5. Momentum Conservation Rules at a Vertex

Let us consider that the four momenta k_1, k_2, k_3, k_4, arriving at a given vertex v belong to the support of the four sectors σ_1, σ_2, σ_3, σ_4, in slices i_1, i_2, i_3, i_4. In Fourier space the vertex (2.6) implies constraints on the momenta, after the infinite volume limit has been taken. Each spatial component of the sum of the four momenta must be an integer multiple of 2π in the initial basis, and the sum of the four Matsubara frequencies must also be zero.

In our tilted basis (e_+, e_-), this translates into the conditions:

$$k_{1,0} + k_{2,0} + k_{3,0} + k_{4,0} = 0, \tag{2.29}$$

$$k_{1,+} + k_{2,+} + k_{3,+} + k_{4,+} = 2n_+, \tag{2.30}$$

$$k_{1,-} + k_{2,-} + k_{3,-} + k_{4,-} = 2n_-, \tag{2.31}$$

where n_+ and n_- must have identical parity.

We want to rewrite the two last equations in terms of the fractional parts q_1, q_2, q_3 and q_4.

Since an even sum of integers which are ± 1 is even, we find that (2.30) and (2.31) imply

$$q_{1,+} + q_{2,+} + q_{3,+} + q_{4,+} = 2m_+, \tag{2.32}$$

$$q_{1,-} + q_{2,-} + q_{3,-} + q_{4,-} = 2m_-, \tag{2.33}$$

with m_+ and m_- integers. Let us prove now that except in very special cases, these integers must be 0. Since $|q_{j,\pm}| \leqslant 1$, $|m_\pm| \leqslant 2$. But $|q_{j,\pm}| = 1$ is possible only for $s_{j,\pm} = 0$. Therefore $|m_\pm| = 2$ implies $s_{j,\pm} = 0$, $\forall j$. Now suppose e.g., $|m_+| = 1$. Then $s_{j,+}$ is 0 for at least two values of j. Indeed for

$s_{j,\pm} \neq 0$ we have $|q_{j,\pm}| \leqslant \sqrt{2}\, M^{-1}$, and assuming $3\sqrt{2}\, M^{-1} < 1$, Eq. (2.32) could not hold.

We have therefore proved

Lemma 2. $m_+ = 0$ unless $s_{j,+}$ is 0 for at least two values of j, and $m_- = 0$ unless $s_{j,-}$ is 0 for at least two values of j.

Let us analyze in more detail equations (2.32) and (2.33) for $|m_+| = |m_-| = 0$. Consider e.g., (2.32). By a relabeling we can assume without loss of generality that $s_{1,+} \leqslant s_{2,+} \leqslant s_{3,+} \leqslant s_{4,+}$ Then either $s_{1,+} = i_1$ or $s_{1,+} < i_1$, in which case combining equations (2.32) and (2.28) we must have:

$$3\sqrt{2}\, M^{-s_{2,+}} \geqslant 2M^{-s_{1,+}}/\pi M, \tag{2.34}$$

which means

$$s_{2,+} \leqslant s_{1,+} + 1 + \frac{\log(3\pi/\sqrt{2})}{\log M}. \tag{2.35}$$

This implies

$$|s_{2,+} - s_{1,+}| \leqslant 1 \tag{2.36}$$

if $M > 3\pi/\sqrt{2}$, which we assume from now on.

The conclusion is:

Lemma 3. If $m_\pm = 0$, either the smallest index $s_{1,\pm}$ coincides with its scale i_1, or the two smallest indices among $s_{j,\pm}$ differ by at most one unit.

Now we can summarize the content of both lemmas in a slightly weaker but simpler lemma:

Lemma 4. (A) (Single Slice Case). The two smallest indices among $s_{j,+}$ for $j = 1, 2, 3, 4$ differ by at most one unit, and the two smallest indices among $s_{j,-}$ for $j = 1, 2, 3, 4$ differ by at most one unit.

(B) Multislice Case. The two smallest indices among $s_{j,+}$ for $j = 1, 2, 3, 4$ differ by at most one unit or the smallest one, say $s_{1,+}$ must coincide with its scale i_1, which must then be strictly smaller than the three other scales i_2, i_3 and i_4. Exactly the same statement holds independently for the minus direction.

3. THE EXPANSION

For simplicity let us prove the theorem for the pressure:

$$p = \lim_{V \to \infty} \frac{1}{|V|} \log Z(V), \tag{3.1}$$

$$Z(V) = \int d\mu_{C_i}(\bar{\psi}, \psi)\, e^{\lambda \int_V d^3x (\sum_a \bar{\psi}\psi)^2 (x)} \tag{3.2}$$

where $d\mu_{C_i}(\bar{\psi}, \psi)$ is the Grassmann Gaussian measure of covariance C_i. (The proof extends without difficulty to any Schwinger function at fixed external momenta).

We develop each field and antifield into a sum over sectors, obtaining a collection of sectors $\{\sigma\}$. For each vertex j there are four field or antifields, hence four sectors called σ_j^1, σ_j^2, σ_j^3 and σ_j^4. Integrating over the Grassmann measure, $Z(V)$ becomes:

$$Z(V) = \sum_n \frac{\lambda^n}{n!} \int_{V^n} d^3x_1 \cdots d^3x_n \sum_{a_j, b_j} \sum_{\{\sigma\}} \begin{Bmatrix} x_{1, a_1, \sigma_1^1} & x_{1, b_1, \sigma_1^2} & \cdots & x_{n, a_n, \sigma_n^1} & x_{n, b_n, \sigma_n^2} \\ x_{1, a_1, \sigma_1^3} & x_{1, b_1, \sigma_1^4} & \cdots & x_{n, a_n, \sigma_n^3} & x_{n, b_n, \sigma_n^4} \end{Bmatrix}$$

where we used Cayley's notation for the determinants:

$$\begin{Bmatrix} u_{j, a, \sigma} \\ v_{k, b, \sigma'} \end{Bmatrix} = \det(\delta_{ab} \delta_{\sigma\sigma'} C_{i, \sigma}(u_j - v_k)) \tag{3.3}$$

and a_j, b_j are the spin indices.

We know that if we expand the Cayley determinant the pressure is given by the sum over all connected vacuum graphs with one particular vertex fixed at the origin (using translation invariance). But this formula is not suited for convergence. Instead we want to connect the vertices of the connected vacuum graphs by a tree formula, because these formulas together with Gram's inequality on the remaining determinant are the most convenient to prove convergence.[15, 20] However we want this formula ordered with respect to increasing values of the depth index $l = s_+ + s_- - i + 2$ (which runs between 0 and $i+2$), so that tree lines with lowest depth are expanded first. This is conveniently done using the Taylor jungle formula [ref. 16, Theorem IV.3]. We obtain:

$$p = \sum_n p_n \lambda^n \tag{3.4}$$

$$p_n = \frac{1}{n!} \sum_{\{a,b,\sigma\}} \sum_{\mathcal{J}} \epsilon(\mathcal{J}) \prod_{j=1}^{n} \int dx_j \, \delta(x_1)$$
$$\times \prod_{\ell \in \mathcal{T}} \int_0^1 dw_\ell \, C_{i, a_\ell, \sigma_\ell}(x_\ell, \bar{x}_\ell) \det{}_{\text{left}}(C_{i,\sigma}(w)) \tag{3.5}$$

where $\mathcal{J} = (\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{i+1} \subset \mathcal{F}_{i+2} = \mathcal{T})$ is a layered object called a jungle in ref. 16), $\epsilon(\mathcal{J})$ being an inessential sign. Such a jungle is an increasing sequence of forests $\mathcal{F}_j$. A forest is simply a set of lines which do not make loops, hence in contrast with a tree it can possibly have several connected components. Here the sum is constrained over the jungles whose last layer $\mathcal{F}_{i+2} = \mathcal{T}$ must be a real tree $\mathcal{T}$ connecting the n vertices. This constraint arises because we are computing a connected function, namely the pressure. The notation $\det_{\text{left}}(C_i(w))$ means the determinant made of the fields and antifields left after extraction of the tree propagators. It is therefore a $n+1$ by $n+1$ square matrix of the Cayley type similar to (3.3), but with an additional multiplicative parameter depending on the interpolating parameters $\{w\}$. More precisely, its (f, g) entry between field f and antifield g is zero unless the spin and sectors for f and g coincide. In that case let $\chi(f, j)$ be 1 if field f hooks to vertex j and zero otherwise. Let also l_f be the depth of a field or antifield f. The (f, g) entry of the determinant left in (3.5) is then:

$$C_i(w)_{fg} = \delta_{\sigma(f)\,\sigma(g)} \delta_{a(f)\,b(g)} \sum_{j=1}^{n} \sum_{k=1}^{n} \chi(f, j)\, \chi(g, k)$$
$$\times w^{\mathcal{J}, l_f}(j, k)(\{w\})\, C_{i, a(f), \sigma(f)}(x_j, x_k) \tag{3.6}$$

where $w^{\mathcal{J}, l}(j, k)(\{w\})$ is given by a rather complicated formula:

— If the vertices j and k are not connected by $\mathcal{F}_l$, then $w^{\mathcal{J}, l}(j, k)(\{w\}) = 0$

— If the vertices j and k are connected by $\mathcal{F}_{l-1}$, then $w^{\mathcal{J}, l}(j, k)(\{w\}) = 1$

— If the vertices j and k are connected by $\mathcal{F}_l$, but not by $\mathcal{F}_{l-1}$, then $w^{\mathcal{J}, l}(j, k)(\{w\})$ is the infimum of the w_ℓ parameters for ℓ in the unique path in the reduced forest $\mathcal{F}_l / \mathcal{F}_{l-1}$ connecting the two vertices.[16] [4] The natural convention is that $\mathcal{F}_{-1} = \emptyset$ and that $w^{\mathcal{J}, l}(j, j) = 1$.

[4] The reduced forest $\mathcal{F}_l / \mathcal{F}_{l-1}$ is as usual the one in which all the connected components of $\mathcal{F}_{l-1}$ have been contracted to a single vertex.

We will only need to know that the matrix $w^{\mathcal{J},l}(j,k)(\{w\})$ is a positive n by n matrix with entries labeled by the *vertices* j and k. This is enough to bound these interpolation parameters by 1 in Gram's bound for the $\det_{\text{left}}(C_i(w))$. This is explained in detail in refs. 15 and 18.

Now at any given level l the forest $\mathcal{F}_l$ defines a certain set of $c(l)$ different connected components (some of them eventually reduced to a trivial isolated vertex). To each such connected component one can associate an object G_l^k, $k=1,2,\ldots,c(l)$, which has a well defined number of internal vertices and a well defined even number of external fields $e(G_l^k)$. These external fields are the fields of index greater than l hooked to the internal vertices, which are themselves joined together by the forest $\mathcal{F}_l$. In addition to the internal tree connecting the internal vertices this object contains a set of internal fields still forming a determinant. Therefore G_l^k is not exactly a subgraph, since expanding the determinant would create a collection of subgraphs in the ordinary sense. It is nevertheless a kind of generalization of the notion of subgraph. The connected components G_l^k play a fundamental role in any multislice analysis;[17] their inclusion relations form an other tree, the so-called Gallavotti–Nicolò tree.

Remark that the final tree $\mathcal{T}$ plus the collection $\{\sigma\}$ of sectors for all fields obviously determine the full layered tree structure $\mathcal{J}$ and the connected components G_l^k at level l, together with their number of external legs $e(G_l^k)$. Hence the sums over a, b, σ and $\mathcal{J}$ in (3.5) are redundant, and can be replaced by a simpler sum over a, b, σ and $\mathcal{T}$.

Anticipating on what follows, the power counting of the two point connected components G_l^k (those for which $e(G_l^k)=2$) is marginal. We will need to identify the pair of external fields of these components also called "bipeds," in order to take into account their extra momentum conservation rule. The bipeds b (together with the full final graph which we call G) form a tree for the inclusion relation, called $\mathcal{B}$. This tree is not exactly the "clustering tree structure"[6] or "Gallavotti–Nicolò" tree, whose nodes are the connected components G_l^k and whose lines depict their inclusion relation (the Gallavotti–Nicolò tree is not to be confused with $\mathcal{T}$). It is rather a subtree, in the sense that it is made of those nodes of the "Gallavotti–Nicolò" tree that correspond to bipeds, together with the lines which represent their inclusion relations. We include also in $\mathcal{B}$ the bare vertices from which each biped is made on, and we picture it as follows:

Every biped is pictured as a cross, every bare vertex as a dot. There is an inclusion line from each dot to the smallest biped containing it, and from each biped to the unique next bigger biped containing it. These inclusion lines which form the forest $\mathcal{B}$ are pictured as downwards arrows in Fig. 2. To recover finally a tree, the last vertex or root, pictured as a box, corresponds to the full graph G which contains all the maximal bipeds and

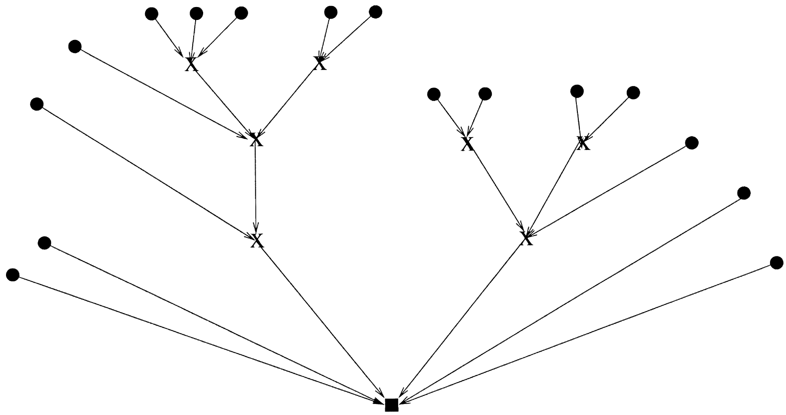

Fig. 2. The forest $\mathscr{B}$.

the remaining dots (in our case of the pressure, it cannot be a biped itself since it is a vacuum graph).

For each biped $b \in \mathscr{B}$ we also fix the two external fields $\bar{\psi}_b$ and ψ_b of the biped. They must be hooked to two vertices in b, v_b and $\bar{v}_b$ hence to two dots for which the path to the root using the downwards arrows in Fig. 2 passes through b. Remark indeed that $\bar{\psi}_b$ and ψ_b must be hooked to two different vertices, since tadpoles obviously vanish in this theory at half-filling (by the particle-hole symmetry). Obviously also the sectors for the fields $\bar{\psi}_b$ and ψ_b, namely $\bar{\sigma}_b$ and σ_b must have the largest depth index l among the four sectors hooked respectively to v_b and $\bar{v}_b$, otherwise b would not be a connected component G_l^k for some l. By exact momentum conservation, the external momentum of the biped must belong to the support of these two sectors. Hence they must have equal or neighboring indices s_+ and s_-. Also when b varies, the fields $\bar{\psi}_b$ and ψ_b, and also the vertices v_b and $\bar{v}_b$ are all disjoint. This is a more subtle property. It is true because by momentum conservation, a field cannot be an external field for two bipeds at two different scales (since, necessarily, the biggest would be one-particle reducible, and momentum conservation would be violated).[5]

[5] Strictly speaking, since our C_0^∞ cutoffs have some overlap, this is true only if we define a biped as a component G_l^k with the external scale at least equal to the maximal internal scale plus 2 (not plus 1), that is with a strict gap between internal and external scales. This inessential complication is left to the reader. The two point connected components without such a strict gap do not create any divergence at all. They can be treated therefore as ordinary connected components, with more than two external legs, in the power counting below.

The set of these data $(\bar{\psi}_b, \psi_b)$ for all $b \in \mathscr{B} - \{G\}$ is denoted $\mathscr{E}\mathscr{B}$. By the previous remarks, it can be described as an even set V (the external vertices of the bipeds), plus a partition of this set into pairs v_b and $\bar{v}_b$, one for each b, and, again for each b, the choice of one field ψ_b hooked to v_b and one antifield $\bar{\psi}_b$ hooked to $\bar{v}_b$.

We now fix $\mathscr{B}$, $\mathscr{E}\mathscr{B}$, $\{a, b\}$, and $\mathscr{T}$, and sum over those $\{\sigma\}$ that give rise to these data. The constraint that $\{\sigma\}$ give rise to these data is indicated by a prime in the corresponding sum. Remark in particular the constraint that for any $b \in \mathscr{B}$ the depth $l(\psi_b)$ must be maximal among the four depths l_1, l_2, l_3, l_4 of the sectors hooked to v_b and the depth $l(\bar{\psi}_b)$ must be maximal among the four depths l_1, l_2, l_3, l_4 of the sectors hooked to $\bar{v}_b$, otherwise the subgraph b would not appear as a connected component G_l^k.

To take into account the momentum conservation constraints we introduce now for each vertex the function $\chi_j(\sigma) = \chi(\sigma_j^1, \sigma_j^2, \sigma_j^3, \sigma_j^4)$ which is 1 if the condition of Lemma 4 is satisfied and 0 otherwise. We introduce also for each two-point subgraph b of the forest $\mathscr{B}$ the constraint $\chi_b(\sigma)$ that states that the sectors of its two external legs σ_b and $\bar{\sigma}_b$ must overlap, that is must be equal or nearest neighbors. These insertions are free since the contributions for which these χ functions are not 1 are zero. They must be done before taking the Gram bound, which destroys the Fourier oscillations responsible for momentum conservation at each vertex. We get:

$$p_n = \frac{1}{n!} \sum_{\substack{\mathscr{B}, \mathscr{E}\mathscr{B} \\ \{a, b\}, \mathscr{T}}} \sum_{\{\sigma\}}' \epsilon(\mathscr{T}) \prod_{j=1}^{n} \int dx_j\, \delta(x_1) \prod_{\ell \in \mathscr{T}} \int_0^1 dw_\ell\, C_{i, a_\ell, \sigma_\ell}(x_\ell, \bar{x}_\ell)$$
$$\times \prod_{j=1}^{n} \chi_j(\sigma) \prod_{b \in \mathscr{B}} \chi_b(\sigma) \det{}_{\text{left}}(C_i(w)). \tag{3.7}$$

We apply now Gram's inequality to the determinant as explained in detail in ref. 18. For that purpose we rewrite $C_{i,\sigma}$ as a product of two half propagators in Fourier space. Taking the square root of the positive matrix w we obtain the bound:

$$\det{}_{\text{left}}(C_i(w)) \leqslant c^n \prod_{f \text{ left}} M^{-(i+l_f)/2} \tag{3.8}$$

where the product runs over all fields and antifields left by the tree expansion. Indeed the half-propagators corresponding to $C_{i,\sigma}$ may be chosen to contribute each to one half of the full propagator scaling factor M^{-i-l_f} in (2.21).

We can now integrate over the positions of the vertices save the fixed one x_1 using the Gevrey scaled decay (2.21) and obtain a bound on the nth order of perturbation theory

$$|p_n| \leqslant \frac{c^n}{n!} M^{-2i} \sum_{\substack{\bar{\mathscr{B}}, \mathscr{EB} \\ \{a,b\}, \mathscr{T}}} \sum_{\{\sigma\}}' \prod_{j=1}^{n} \chi_j(\sigma) \prod_{b \in \mathscr{B}} \chi_b(\sigma) \prod_{\ell \in \mathscr{T}} M^{l_\ell} \prod_{f} M^{-l_f/2} \tag{3.9}$$

where the product over f now runs over all the $4n$ fields and antifields of the theory.

We can check by induction that:

$$\prod_{f} M^{-l_f/2} = \prod_{l=0}^{i+2} \prod_{k} M^{-e(G_l^k)/2}, \tag{3.10}$$

$$\prod_{\ell \in \mathscr{T}} M^{l_\ell} = M^{-i-3} \prod_{l=0}^{i+2} \prod_{k} M^{1}. \tag{3.11}$$

(to prove the last equality, remember that $\mathscr{T}$ is a subtree in each connected component G_l^k). We obtain the bound

$$|p_n| \leqslant \frac{c^n}{n!} M^{-2i} \sum_{\substack{\bar{\mathscr{B}}, \mathscr{EB} \\ \{a,b\}, \mathscr{T}}} \sum_{\{\sigma\}}' \prod_{j=1}^{n} \chi_j(\sigma) \prod_{b \in \mathscr{B}} \chi_b(\sigma) \prod_{l=0}^{i+2} \prod_{k} M^{1-e(G_l^k)/2}. \tag{3.12}$$

Therefore we have exponential decay in index space except for the bipeds $b \in \mathscr{B}$. Indeed for $e \geqslant 4$ we have $e/2-1 \geqslant e/4$. At each vertex j we have four sectors with depths l_j^1, l_j^2, l_j^3 and l_j^4. The data in $\mathscr{EB}$ in particular contain the information about the set V of vertices for which the line with maximal index, l_j^4, is the external line of a biped in $\mathscr{B}$. Therefore we obtain:

$$|p_n| \leqslant \frac{c^n}{n!} M^{-2i} \sum_{\substack{\bar{\mathscr{B}}, \mathscr{EB} \\ \{a,b\}, \mathscr{T}}} \sum_{\{\sigma\}}' \prod_{j=1}^{n} \chi_j(\sigma) \prod_{j \notin V} M^{-(l_j^1+l_j^2+l_j^3+l_j^4)/4} \prod_{j \in V} M^{-(l_j^1+l_j^2+l_j^3)/4}. \tag{3.13}$$

Now we need the following lemma:

Lemma 5. Suppose the four sectors $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ have depths l_1, l_2, l_3 and l_4. Then for fixed σ_4

$$\sum_{\sigma_1, \sigma_2, \sigma_3} \chi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)\, M^{-(l_1+l_2+l_3)/4} \leqslant c.i. \tag{3.14}$$

Proof. Let us say that σ_j collapses with σ_k in the $\pm$ direction, and let us write $\sigma_j \simeq {}_{\pm}\sigma_k$ if $|s_{\pm,j} - s_{\pm,k}| \leqslant 1$. The function χ ensures two collapses, one in each direction, for pairs with minimal values of the corresponding s indices. So it ensures that $\sigma_j \simeq {}_{+}\sigma_k$ and $\sigma_{j'} \simeq {}_{-}\sigma_{k'}$ for some $j \neq k$ and $j' \neq k'$. Now let us make three remarks:

(a) Since $l = s_+ + s_- - i + 2 \geqslant 0$, for a given sector summing over s_+ knowing s_- or vice versa can be done at the cost of a constant, using only a fraction of the decay $M^{-l/4}$.

(b) When a pair j, k collapses in any direction, one element of the pair, say k is not the fixed sector ($k \neq 4$). Using remark a, for fixed σ_j we can sum over σ_k at the cost of a constant, using only a fraction of the decay $M^{-l_k/4}$.

(c) If a sector σ_m does not collapse with any other sector in any direction, we must have some sector j which collapses in both directions with an other sector. This means that $s_{+,m} \geqslant s_{+,j}$ *and* $s_{-,m} \geqslant s_{-,j}$. But then $l_m \geqslant l_j$. If $m \neq 4$ we have therefore

$$M^{-l_m/4} = M^{-(l_m - l_j)/4} M^{-l_j/4} \leqslant M^{-[(s_{+,m} - s_{+,j}) + (s_{-,m} - s_{-,j})]/4}, \tag{3.15}$$

and we can sum over σ_m knowing σ_j again at the cost of a constant using a fraction of the decay $M^{-l_m/4}$.

Putting these three remarks together, we obtain the lemma. Indeed by remark c the eventual sums over sectors which collapse with no other ones cost only constants. The sums over sectors that under collapse relations are connected to the fixed sector σ_4 also cost nothing by remark b. Finally there can remain at most one non trivial equivalence class under collapse which does not contain the fixed sector σ_4. We pay a single factor i to fix say some $\sigma_{j,+}$ within this class, and using again Remarks a and b we can achieve all other sums in that class paying only some constants. ∎

Now to prove the theorem we return to (3.13).

We can use the same strategy as in refs. 6 and 7 to sum over sectors, following the natural ordering from leaves to root of our tree $\mathcal{T}$. Every vertex is related to the root vertex by a single path in the tree which starts by a well defined half-line hooked to that vertex called the root half-line. We will pay for the sector sum at that vertex keeping the sector of that root half line fixed. This last sector will be fixed later when the vertex bearing the other half of the tree line associated to the root half-line is considered. For the root vertex (of a vacuum graph) there will be still one last sector to fix. By Lemma 5, each sum over the sectors of a regular vertex (not in V)

costs therefore only $c.i$. Now from the data in $\mathscr{E}\mathscr{B}$ we know how the vertices in V group into pairs $\bar{v}_b, v_b$ associated to the bipeds b, and by the momentum conservation constraint χ_b we know that $\sigma^4_{\bar{v}_b} \simeq \sigma^4_{v_b}$ and $l^4_{\bar{v}_b} \simeq l^4_{v_b}$. Moreover we can choose the root to be external for a *maximal* biped b_0 of $\mathscr{B}$, so that either $\bar{v}_b$ or v_b is the root (if $b=b_0$), or $\sigma^4_{\bar{v}_b}$ or $\sigma^4_{v_b}$ is a root half-line (because the root, being outside, is either left or right of the two point subgraph). Suppose the root half-line is $\sigma^4_{\bar{v}_b}$. We can now "cut" one line of the path P_b which joins in the tree $\mathscr{T}$ these two external vertices of $b/\mathscr{B}$ (the reduced graph in which maximal subbipeds of $\mathscr{B}$ inside b have been collapsed to a single point). This arbitrary cut simply means that for that tree line we forget the constraint that identifies the sectors of the two corresponding half-lines. (This certainly only enlarges the sum in (3.13)). We can now follow the same summation process as before, but separately for the two halfs of the graph b which are on both side of the cut. For the part on the side of $\sigma^4_{\bar{v}_b}$ we sum towards that half-line and for the other part we sum towards the half-line $\sigma^4_{v_b}$ as if it was a new root. Finally since $\sigma^4_{\bar{v}_b} \simeq \sigma^4_{v_b}$, we can sum over the sectors for both $\bar{v}_b$ and v_b at once using Lemma 5 and we get

$$\sum_{\substack{\sigma^4_{\bar{v}_b}\text{ fixed}, \sigma^4_{\bar{v}_b} \simeq \sigma^4_{v_b} \\ \sigma^1_{\bar{v}_b}, \sigma^2_{\bar{v}_b}, \sigma^3_{\bar{v}_b}, \sigma^1_{v_b}, \sigma^2_{v_b}, \sigma^3_{v_b}}} \chi_{\bar{v}_b}(\sigma)\, \chi_{v_b}(\sigma)\, M^{-(1/4)(l^1_{\bar{v}_b}+l^2_{\bar{v}_b}+l^3_{\bar{v}_b}+l^1_{v_b}+l^2_{v_b}+l^3_{v_b})} \leqslant (c.i)^2. \tag{3.16}$$

We have finally to pay for summing over the last root sector. When the root vertex is not in V, we have a last sum to perform over some σ^4_j but we can use the decay factor $M^{-l^4_j}$ in (3.13) to pay for it. So this last sum costs an additional factor i. But when the root is say v_{b_0}, there is no $M^{-l^4_{v_b}}$ decay in (3.13) and this last sum therefore costs not a factor i but a factor i^2.

Hence we arrive at:

$$|p_n| \leqslant M^{-2i} i^{n+2} \sum_{\mathscr{B}, \mathscr{E}\mathscr{B}, \{a,b\}, \mathscr{T}} \frac{c^n}{n!}. \tag{3.17}$$

The sum over spin indices $\{a, b\}$ trivially costs at most 4^n, so from now on let us work with fixed $\{a, b\}$. But to bound the sum over $\mathscr{B}$, $\mathscr{E}\mathscr{B}$ and $\mathscr{T}$, one has to exploit the fact that there is a balance: roughly speaking for $\mathscr{B}$ small the sum over $\mathscr{B}$, $\mathscr{E}\mathscr{B}$ does not cost much and we have Cayley's theorem which states that the number of possible trees $\mathscr{T}$ at order n is n^{n-2}; for $\mathscr{B}$ large (many bipeds) the choice of which vertices belong to which biped may be costly, but once it is done, the compatible trees are much fewer. This is captured in the following lemma:

Lemma 6. There exists some constant c such that

$$\sum_{\mathscr{B}, \mathscr{E}\mathscr{B}, \mathscr{T}} \frac{1}{n!} \leqslant c^n. \tag{3.18}$$

Proof. Recall that c is our generic name for a constant. Let us call $N_b = |\mathscr{B}|$ the total number of bipeds. For each $b \in \mathscr{B}$ (including the box G) let us call d_b the number of links in $\mathscr{B}$ whose down end is b and $n_b \leqslant d_b$ the number of bare vertices that belong to b and no smaller biped (dots in Fig. 2 with a down link ending at b). Remark that $\sum_b n_b = n$ since each dot belongs to some element of $\mathscr{B}$ (since we included G in $\mathscr{B}$). Moreover $N_b \leqslant n/2$ and $\sum_b d_b = n + N_b - 1 \leqslant 2n$. Therefore paying c^n we can fix N_b and the numbers n_b and d_b for each b.

We perform inductively the counting over the cardinal of the set $(\bar{\mathscr{B}}, \mathscr{E}\mathscr{B}, \mathscr{T})$ starting from the leaves in Fig. 2 towards the root. To choose the n_b vertices in each biped we have to pay $n!/\prod_b n_b!$. To build the tree $\mathscr{T}$ we build its restriction to each reduced element of $\mathscr{B}$, which contains d_b vertices (n_b ordinary four point vertices and $d_b - n_b$ reduced two point vertices).

Now it is well known that there are less than c^n different planar trees obtained by connecting n vertices to a root as in Fig. 2. Let τ be a subset of such trees differing one from the other only through some permutation of the branches. Let B_τ be the family of all $\mathscr{B}$ which can be associated with at least one element of the subset τ.

Since by Cayley's theorem the number of trees on n vertices is n^{n-2} hence bounded by $c^n n!$, the number of possible choices for $\mathscr{T}$ is bounded by $\prod_b c^{d_b} d_b!$, hence by $c^n \prod_b d_b!$. Finally to choose $\mathscr{E}\mathscr{B}$ we fix for each b the two fields ψ_b and $\bar{\psi}_b$. Since as remarked above they must be hooked to the n_b vertices that belong to b and no smaller biped, the number of choices for $\mathscr{E}\mathscr{B}$ is bounded by $\prod_b (4n_b)^2$ hence by c^n since $\sum_b n_b = n$. Multiplying all these numbers we obtain a bound, but here comes the subtle point: in this way we have counted $\prod_b (d_b - n_b)!$ times each configuration $\mathscr{B}, \mathscr{E}\mathscr{B}, T$. Indeed the tree in Fig. 2 is *unlabeled*. A permutation group with $\prod_b (d_b - n_b)!$ elements acts on it, permuting at each fork b the $d_b - n_b$ maximal bipeds in b, and each permutation on one element of $\mathscr{B}, \mathscr{E}\mathscr{B}, \mathscr{T}$ built in the way described above gives again the same element.

Hence taking these remarks into account we obtain:

$$\sum_{\mathscr{B}, \mathscr{E}\mathscr{B}, \mathscr{T}} 1 \leqslant c^n \sup_\tau \sum_{\mathscr{B} \in B_\tau, \mathscr{E}\mathscr{B}, \mathscr{T}} 1 \leqslant c^n n! \prod_b \frac{d_b!}{n_b! \, (d_b - n_b)!} \leqslant c^n n! \tag{3.19}$$

which completes the proof of the lemma. ∎

Proof of Theorem 1. Combining this lemma and (3.17), our final bound is

$$|p_n| \leqslant i^2 M^{-2i}(c.i)^n. \tag{3.20}$$

This achieves the proof of Theorem 1. Remark that for a more general Schwinger function the prefactor $i^2.M^{-2i}$ would be different but this has no influence on the radius of convergence. ▮

4. CONVERGENT CONTRIBUTIONS IN THE MULTISLICE THEORY

To analyze the multislice theory, we remark first that by Lemma 1, integration over a vertex using the decay of a line with indices i and l costs M^{2i+l}. Therefore it is convenient to select the multislice tree for a graph by optimizing over the index $r = I(i+l/2)$, where I is the integer part, so that r remains integer. From now on, we may forget the integer part I which is inessential.

In other words since i in this section is no longer fixed, we define the sectors as triplets $\sigma = (i, s_+, s_-)$, with $1 \leqslant i \leqslant i_{\max}(T)$, $0 \leqslant s_+ \leqslant i$, $0 \leqslant s_- \leqslant i$, and $s_+ + s_- \geqslant i-2$. The depth l of a sector is still $l = s_+ + s_- - i + 2$, and the momentum cutoff u_{si} for a sector is

$$u_\sigma(k_0, k_+, k_-) = u_i[k_0^2 + 4\cos^2(\pi k_+/2)\cos^2(\pi k_-/2)] \\ \times v_{s_+}[\cos^2(\pi k_+/2)]\, v_{s_-}[\cos^2(\pi k_-/2)]. \tag{4.1}$$

The propagator in sector σ is in momentum space

$$C_\sigma(k_0, k_+, k_-) = \frac{u_\sigma(k_0, k_+, k_-)}{ik_0 - 2\cos(\pi k_+/2)\cos(\pi k_-/2)}. \tag{4.2}$$

The full propagator (with u.v. cutoff corresponding to the inessential removal of the slice $i = 0$) is

$$C = \sum_\sigma C_\sigma = \sum_{r=1}^{r_{\max}(T)} C_r, \qquad r_{\max}(T) = 1 + 3i_{\max}(T)/2, \tag{4.3}$$

the rth slice of the propagator being defined as the sum over all sectors with $i + l/2 = r$:

$$C_r = \sum_{\substack{\sigma \\ i(\sigma)+l(\sigma)/2 = r}} C_\sigma = \sum_l C_{r,l},\; C_{r,l} = \sum_{\substack{\sigma \\ i(\sigma)+l(\sigma)/2 = r \\ l(\sigma) = l}} C_\sigma. \tag{4.4}$$

For more generality we shall formulate this time our theorem for Schwinger functions S_{2p} with $p \geqslant 2$. We perform the same Taylor jungle expansion as in the previous section, but with respect to increasing values of the r index, and obtain (omitting the inessential dependence on the external momenta or positions):

$$S_{2p} = \sum_n S_{2p,n} \lambda^n, \tag{4.5}$$

$$S_{2p,n} = \frac{1}{n!} \sum_{\{a,b,\sigma\}} \sum_{\mathcal{J}} \epsilon(\mathcal{J}) \prod_v \int dx_v \prod_{\ell \in \mathcal{T}} \int_0^1 dw_\ell \, C_{i,\sigma_\ell}(x_\ell, y_\ell) \det{}_{\text{left}}(C_{i,\sigma}(w)), \tag{4.6}$$

where $\mathcal{J} = (\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r_{\max}(T)} = \mathcal{T})$.

Knowing all the scales and sectors of all the fields, the connected components G_r^k at each level r again form a Gallavotti–Nicolò tree for the inclusion relations. As seen below, in the r space power counting is standard, namely the bipeds are linearly divergent, and the "quadrupeds," namely the non-trivial[6] components G_r^k in the Gallavotti–Nicolò tree which have $e(G_r^k) = 4$, are marginal. The bipeds require renormalization and will be treated in another paper. In this section we state two theorems: one for the "completely convergent" part of the expansion, that is the one which has neither bipeds nor quadrupeds, and the other for the "biped-free" part of the expansion which has no biped but can have quadrupeds. Indeed the first theorem is easier, so that order of presentation seems more pedagogical.

Therefore we define the structure of all divergent components as $\mathcal{B} \cup \mathcal{Q}$ with inclusion relations exactly as in Fig. 2. $\mathcal{B}$ is the set of bipeds, hence of connected components G_r^k with $e(G_r^k) = 2$ and $\mathcal{Q}$ is the set of quadrupeds, including the full graph G pictured as the box in Fig. 2, which may or may not have four external legs, depending whether $p = 2$ or $p > 2$. As in the previous section we also define $\mathcal{E}\mathcal{Q}$ as the data for the external legs of every quadruped $q \in \mathcal{Q}$.

We organize our sum as in the previous section and get the analog of (3.7)

$$S_{2p,n} = \frac{1}{n!} \sum_{\substack{\mathcal{B}, \mathcal{E}\mathcal{B} \\ \mathcal{Q}, \mathcal{E}\mathcal{Q} \\ \{a,b\}, \mathcal{T}}} \sum_{\{\sigma\}}{}' \epsilon(\mathcal{T}) \prod_v \int dx_v \prod_{\ell \in \mathcal{T}} \int_0^1 dw_\ell \, C_{i,\sigma_\ell}(x_\ell, y_\ell) \det{}_{\text{left}}(C_{i.\sigma}(w)). \tag{4.7}$$

[6] Non-trivial here means "not reduced to a single vertex."

The integration $\int dx_v$ is performed as usually only for the internal vertices. The completely convergent part of the functions S_{2p}, called $S^c_{2p} = \sum_n S^c_{2p,n}\lambda^n$, is now the sum over all contributions for which $\mathscr{B} = \mathscr{Q} = \varnothing$, namely $e(G^k_r) > 4$, $\forall r, k$ (this requires $p \geqslant 3$):

$$S^c_{2p,n} = \frac{1}{n!} \sum_{\substack{\mathscr{B}=\mathscr{Q}=\varnothing \\ \{a,b\}\ \mathscr{T}}} \sum_{\{\sigma\}}{}' \epsilon(\mathscr{T}) \prod_v \int dx_v \prod_{\ell \in \mathscr{T}} \int_0^1 dw_\ell\, C_{i,\sigma_\ell}(x_\ell, y_\ell) \det_{\text{left}}(C_{i,\sigma}(w)). \tag{4.8}$$

We can now state our second result:[7]

Theorem 2. The functions S^c_{2p} are analytic in λ for $|\lambda \log T| \leqslant c$ hence their radius of convergence R_T at temperature T satisfies

$$R_T \geqslant c/|\log T|. \tag{4.9}$$

Proof. Before Gram's bound we now introduce only the momentum conservation constraints at each bare vertex $j = 1,..., n$:

$$S^c_{2p,n} = \frac{1}{n!} \sum_{\substack{\mathscr{B}=\mathscr{Q}=\varnothing \\ \{a,b\}\ \mathscr{T}}} \sum_{\{\sigma\}}{}' \epsilon(\mathscr{T}) \prod_{j=1}^n \chi_j(\{\sigma\})$$

$$\times \prod_v \int dx_v \prod_{\ell \in \mathscr{T}} \int_0^1 dw_\ell\, C_{i,\sigma_\ell}(x_\ell, y_\ell) \det_{\text{left}}(C_{i,\sigma}(w))|_{x_0=0}. \tag{4.10}$$

We can now apply Gram's inequality to the determinant as in the previous section, and integrate again over all positions of the internal vertices (save one in the case of the pressure) using the decay of the tree propagators.

Since $(i+l)/2 = r/2 + l/4$, we obtain, exactly in the same way as (3.9), the bound (holding external vertices fixed):

$$|S^c_{2p,n}| \leqslant \frac{c^n}{n!} \sum_{\substack{\mathscr{B}=\mathscr{Q}=\varnothing \\ \{a,b\}\ \mathscr{T}}} \sum_{\{\sigma\}}{}' \prod_{j=1}^n \chi_j(\{\sigma\}) \prod_{\ell \in \mathscr{T}} M^{2r_\ell} \prod_f M^{-r_f/2 - l_f/4}, \tag{4.11}$$

[7] This definition of S^c_{2p} has the disadvantage not to be cutoff independent. Indeed it includes not only the sum of all the graphs without two and four point subgraphs, which is a cutoff-independent object, but also some part of the amplitudes of graphs which do have such two or four point subgraphs, namely those parts in which these divergent subgraphs do not appear as connected components G^k_r. However this definition is the most natural one in the context of a multiscale expansion, and is similar to those of refs. 6, 12, and 14.

where the product over f runs again over all the fields and antifields of the theory, and the sum over $\{\sigma\}$ is again constrained to be compatible with the data ($\mathcal{B}=\mathcal{Q}=\emptyset$, $\{a,b\}$, $\mathcal{T}$), as indicated by the prime notation.

Using again (3.10) and (3.11) we obtain in a similar way:

$$\prod_f M^{-r_f/2}=\prod_{r=0}^{r_{\max}(T)}\prod_k M^{-e(G_r^k/2)}, \tag{4.12}$$

$$\prod_{\ell\in\mathcal{T}} M^{2r_\ell}=M^{-2r_{\max}(T)-2}\prod_{r=0}^{r_{\max}(T)}\prod_k M^2, \tag{4.13}$$

so that apart from a certain n independent factor that cannot influence the radius of convergence we get:

$$|S^c_{2p,n}|\leqslant\frac{c^n}{n!}\sum_{\substack{\mathcal{B}=\mathcal{Q}=\emptyset\\ \{a,b\},\mathcal{T}}}\sum_{\{\sigma\}}{}^{'}\prod_{j=1}^{n}\left[\chi_j(\{\sigma\})\,M^{-(l_j^1+l_j^2+l_j^3+l_j^4)/4}\right]\prod_{r=0}^{r_{\max}(T)}\prod_k M^{2-e(G_r^k)/2} \tag{4.14}$$

where at a given vertex j we call $l_j^1,\ldots,l_j^4$ the depths of the four sectors hooked to j.

Since for all r,k such that G_r^k is non-trivial, $e(G_r^k)\geqslant 6$, we have $2-e(G_r^k)/2\leqslant -e(G_r^k)/6$ for all such r,k. By a standard argument (see, e.g., ref. 17) we conclude that if $r_j^1\leqslant\cdots\leqslant r_j^4$ are the r scales of the four sectors hooked to j:

$$\prod_{r=0}^{r_{\max}(T)}\prod_k M^{2-e(G_r^k)/2}\leqslant\prod_j M^{-[(r_j^2-r_j^1)+(r_j^3-r_j^1)+(r_j^4-r_j^1)]/6}. \tag{4.15}$$

Therefore:

$$|S^c_{2p,n}|\leqslant\frac{c^n}{n!}\sum_{\{a,b\},\mathcal{T}}\sum_{\{\sigma\}}{}^{'}\prod_{j=1}^{n}\chi_j(\{\sigma\})\,M^{-\sum_{k=1}^4 l_j^k/4-\sum_{k\neq k'}|r_j^k-r_j^{k'}|/18}. \tag{4.16}$$

Now with a fraction (say half) of the decay factor $\prod_j M^{-\sum_{k\neq k'}|r_j^k-r_j^{k'}|/18}$ it is easy to perform the sum over all r indices for all the fields (just follow the tree like in the previous section: at each vertex three r indices can be summed holding the fourth fixed, which is the one of the tree line going towards the root, and iterate until the root). It is also possible with a fraction of the decay factor $M^{-\sum_{k=1}^4 l_j^k/4}$ to sum over the indices i once the indices r have been summed, since $i=r-l/2$. From now on we consider therefore the former i indices as summed, although no longer all equal as in the previous section).

To sum over the sectors $s_j^{\pm}$ once scale indices r and i are fixed, we have to be careful that case B of Lemma 4 must now be used since we are in a multislice case. The new possibility of case B of Lemma 4 is that at a given vertex j we can have $s^1_{j,\pm} = i^1_j < s^k_{j,\pm}$, $k = 2, 3, 4$. Let us say that in this case the vertex j is $\pm$ *special*. In that case, since $s^k_{j,\pm} \leqslant i^k_j \leqslant r^k_j$, we have

$$|s^k_{j,\pm} - s^1_{j,\pm}| \leqslant r^k_j - i^1_j = r^k_j - r^1_j + l^1_j/2 \leqslant |r^k_j - r^1_j| + l^1_j/2. \tag{4.17}$$

It is therefore easy to bound a fraction of the decay factor in (4.16) by the product over the special vertices and directions of another decay factor suitable for the summation of s indices. For instance:

$$\prod_{j \pm \text{special}} M^{-\sum_{k=1}^4 l^k_j/8 - \sum_{k \neq k'} |r^k_j - r^{k'}_j|/36} \leqslant \prod_{j \pm \text{special}} M^{-\sum_{k \neq k'} |s_{k,\pm} - s_{k',\pm}|/108}. \tag{4.18}$$

Using this decay factor it is trivial to sum up all the $s_{\pm}$ indices of a *special* vertex, holding one fixed, namely the one of the tree line going towards the root. The indices in the other direction of the special vertex are easily summed with an other fraction of the l decay factor, namely $M^{-\sum_{k=1}^4 l^k_j/8}$ Finally the sum over indices of the regular vertices which are special neither in the plus nor in the minus direction can be handled exactly as in the previous section, using up the remaining $\prod_{j \, \textit{not} \, \text{special}} M^{-\sum_{k=1}^4 l^k_j/8}$ factor. Indeed their momentum conservation is identical to Case A of Lemma 5. The corresponding sums cost therefore at most $|ci_{\max}(T)|^n$, hence at most $|c \log T|^n$

This achieves the proof of Theorem 2. ∎

Returning to (4.7), we define the biped-free part of the functions S_{2p}, called $S^{bf}_{2p} = \sum_n S^{bf}_{2p,n} \lambda^n$, as the sum over all contributions for which $\mathscr{B} = \varnothing$ namely $e(G^k_r) > 2$, $\forall r, k$ (this requires $p \geqslant 2$):

$$S^{bf}_{2p,n} = \frac{1}{n!} \sum_{\substack{\mathscr{B} = \varnothing \\ \mathscr{Q}, \mathscr{E}\mathscr{Q}, \{a,b\}\, \mathscr{T}}} \sum_{\{\sigma\}}{}' \epsilon(\mathscr{T}) \prod_v \int dx_v \prod_{\ell \in \mathscr{T}} \int_0^1 dw_\ell \, C_{i,\sigma_\ell}(x_\ell, y_\ell) \det{}_{\text{left}}(C(w)). \tag{4.19}$$

We can now state our third result:[8]

Theorem 3. The functions S^{bf}_{2p} are analytic in λ for $|\lambda \log^2 T| \leqslant c$ hence their radius of convergence R_T at temperature T satisfies

$$R_T \geqslant c/|\log^2 T|. \tag{4.20}$$

[8] This result involves again a cutoff-dependent quantity, S^{bf}_{2p}, but with a little additional care it should be possible to prove it also for a cutoff independent quantity, namely the sum of all skeleton graphs. Indeed we know that extracting the self energy part of the theory can be done constructively, at the cost of a slightly more complicated expansion than a simple tree expansion (see ref. 7, Appendix B).

Proof. Before Gram's bound we now introduce not only the momentum conservation constraints at each bare vertex $j=1,..., n$ but also for each quadruped $q \in \mathcal{Q}$:

$$S^{bf}_{2p,n} = \frac{1}{n!} \sum_{\substack{\mathcal{Q}, \mathcal{E}\mathcal{Q} \\ \{a,b\}, \mathcal{T}}} \sum_{\{\sigma\}}' \epsilon(\mathcal{T}) \prod_{j=1}^{n} \chi_j(\{\sigma\}) \prod_q \chi_q(\{\sigma\})$$

$$\times \prod_v \int dx_v \prod_{\ell \in \mathcal{T}} \int_0^1 dw_\ell \, C_{i,\sigma_\ell}(x_\ell, y_\ell) \det\nolimits_{\text{left}} (C(w))|_{x_0=0}. \tag{4.21}$$

We can now apply Gram's inequality to the determinant as in the previous section, and integrate again over all positions of the vertices save one using the decay of the tree propagators. We obtain the analog of (4.14):

$$|S^{bf}_{2p,n}| \leqslant \frac{c^n}{n!} \sum_{\substack{\mathcal{Q}, \mathcal{E}\mathcal{Q} \\ \{a,b\}, \mathcal{T}}} \sum_{\{\sigma\}}' \prod_{q \in \mathcal{Q}} \chi_q(\{\sigma\})$$

$$\times \prod_{j=1}^{n} \chi_j(\{\sigma\}) \, M^{-(l_j^1+l_j^2+l_j^3+l_j^4)/4} \prod_{r=0}^{r_{\max}(T)} \prod_k M^{2-e(G_r^k)/2}. \tag{4.22}$$

We have no longer complete exponential decay between the scales of the legs of any vertex. But the only missing piece corresponds to the quadrupeds, for which in (4.22) the factor $2-e(G_r^k)/2$ is zero. This suggests an inductive bound which works inside each reduced component $q/\mathcal{Q}$. The necessary data to perform this analysis are given in $(\mathcal{Q}, \mathcal{E}\mathcal{Q})$. Like in the previous section, let us introduce n_q and d_q as the number of ordinary vertices and the total number of vertices in the reduced component $q/\mathcal{Q}$, so that $\sum_{q \in \mathcal{Q}} n_q = n$ and $\sum_{q \in \mathcal{Q}} (d_q - n_q) = |\mathcal{Q}| \leqslant n-1$.[9] Let us fix the largest scale r_q inside q. Because the momentum conservation constraints for the external legs of q are included in (4.22), the sums over r scales inside every reduced component $q/\mathcal{Q}$ (including the last one G corresponding to the box in Fig. 2) can be performed exactly like in the previous paragraph, at a cost of c^{d_q} using the line with scale r_q as root for the r indices summation. Similarly the sums over the $s_\pm$ internal indices could be easily bounded by $c^{d_q} |\log T|^{d_q}$, using any given sector of an external line of q as a root for these summations. But this bound is not optimal. Let us prove that we can do better and perform these sums at a cost of only $c^{d_q} |\log T|^{d_q - 1}$,

[9] The fact that any forest of quadrupeds has at most $n-1$ elements is a rather obvious statement, proved for instance in ref. 21, Lemma C1).

holding all four external sectors of the quadruped fixed. By remark c in Lemma 5 and the analysis above, we pay a $|\log T|$ factor only for the vertices with two disjoint collapsing pairs. If one internal vertex of q or the external legs of q do not have disjoint collapsing pairs, we gain directly the necessary $|\log T|$ factor for q later in the analysis. Otherwise, following the tree towards the root of the quadruped, like in Section 3, we pay only at most $|\log T|^{d_q-1}$, because there is at least one sector sum fixed by the external data *in addition* to the root: it is the one corresponding to the collapsing pair of the external legs of q *not* containing the root.[10] This proves the improved bound $c^{d_q}\,|\log T|^{d_q-1}$ for the $s_\pm$ internal summations.

Now for each q we have also in addition to pay a single additional $|\log T|$ factor to fix the scale r_q. Multiplying all these factors, we get for fixed $(\mathcal{Q}, \mathcal{EQ}, \mathcal{T})$:

$$\sum_{\{\sigma\}}{}' \prod_{q\in\mathcal{Q}} \chi_q(\{\sigma\}) \prod_{j=1}^{n} \chi_j(\{\sigma\})\, M^{-(l_j^1+l_j^2+l_j^3+l_j^4)/4} \prod_{r=0}^{r_{\max}(T)} \prod_{k} M^{2-e(G_r^k)/2}$$

$$\leqslant \prod_q c^{d_q}\,|\log\ T|^{d_q} \leqslant c^n\,|\log T|^{2n-1}. \tag{4.23}$$

This completes the proof of Theorem 3, modulo the analog of Lemma 6:

Lemma 7. There exists some constant c such that

$$\sum_{\substack{\mathcal{Q},\,\mathcal{EQ}\\ \{a,b\}\ \mathcal{T}}} \frac{1}{n!} \leqslant c^n. \tag{4.24}$$

Proof. The proof is identical to the one of Lemma 6, except for the little change that a given leg can now be external to *several* quadrupeds. It is easy to take care of this detail: in the sum over $\mathcal{EQ}$ there is simply a factor d_q^4 instead of n_b^2. But since $\sum_q d_q \leqslant 2n$, it is again bounded by c^n. ∎

We expect the radius of analyticity for the full theory (with bipeds) at temperature T to satisfy the same bound as Theorem 3. Indeed thanks to particle-hole symmetry, at half-filling the square Fermi-surface is preserved

[10] We remark that disjoint collapsing pairs at a vertex correspond exactly to the combinatoric of a ϕ^4 vector model. We know that for a Feynman graph we would pay in fact $|\log T|^{cc}$ where cc is the number of closed cycles. It is well known that this number for a quadruped with d vertices is at most $d-1$. Our argument is a slight adaptation of this fact, necessary because we know only a spanning tree, not the exact loop structure and closed cycles of a quadruped.

under the RG flow. In contrast with the jellium case, there is therefore no need to include any counterterm to formulate the analyticity theorem for the full theory with bipeds. Nevertheless power counting must be improved, i.e., one has to transfer some internal convergence to the external legs, hence to prove that by some Ward identity, the apparently divergent two point contributions are really convergent. This is postponed to a future paper.

ACKNOWLEDGMENTS

We thank F. Bonetto for many discussions and common work on a preliminary version of this paper. In particular he found the elegant definition of sectors in Section 2.2. We also thank C. Kopper for a critical reading of the manuscript, and one of our referee's for his excellent job.

REFERENCES

1. G. Benfatto and G. Gallavotti, Perturbation theory of the Fermi surface in a quantum liquid. A general quasi particle formalism and one dimensional systems, *J. Statist. Phys.* **59**:541 (1990).
2. J. Feldman and E. Trubowitz, Perturbation theory for many fermion systems, *Helv. Phys. Acta* **63**:156 (1991).
3. J. Feldman and E. Trubowitz, The flow of an electron–phonon system to the superconducting state, *Helv. Phys. Acta* **64**:213 (1991).
4. G. Benfatto, G. Gallavotti, A. Procacci, and B. Scoppola, *Commun. Math. Phys.* **160**:93 (1994).
5. F. Bonetto and V. Mastropietro, *Commun. Math. Phys.* **172**:57 (1995).
6. M. Disertori and V. Rivasseau, Interacting Fermi liquid in two dimensions at finite temperature, Part I: Convergent attributions, *Comm. Math. Phys.* **215**:251, (2000).
7. M. Disertori and V. Rivasseau, Interacting Fermi liquid in two dimensions at finite temperature, Part II: Renormalization, *Commun. Math. Phys.* **215**:291 (2000).
8. M. Salmhofer, Continuous renormalization for Fermions and Fermi liquid theory, *Commun. Math. Phys.* **194**:249 (1998).
9. J. Feldman, H. Knörrer, D. Lehmann, and E. Trubowitz, Fermi liquids in two space time dimensions, in *Constructive Physics*, V. Rivasseau, ed., Springer Lectures Notes in Physics, Vol. 446 (1995).
10. J. Feldman, M. Salmhofer, and E. Trubowitz, Perturbation theory around non-nested Fermi surfaces ii. Regularity of the moving fermi surface, rpa contributions, *Comm. Pure. Appl. Math.* **51**:1133 (1998); Regularity of the moving fermi surface, the full selfenergy, to appear in *Comm. Pure. Appl. Math.*
11. M. Salmhofer, Improved power counting and fermi surface renormalization, *Rev. Math. Phys.* **10**:553 (1998).
12. J. Feldman, J. Magnen, V. Rivasseau, and E. Trubowitz, An infinite volume expansion for many fermion Green's functions, *Helv. Phys. Acta* **65**:679 (1992).
13. J. Magnen and V. Rivasseau, A single scale infinite volume expansion for three dimensional many fermion Green's functions, *Math. Phys. Electron. J.*, Vol. 1 (1995).

14. M. Disertori, J. Magnen, and V. Rivasseau, Interacting Fermi liquid in three dimensions at finite temperature: Part I: Convergent contributions, *Ann. H. Poincaré* **2** (2001).
15. A. Abdesselam and V. Rivasseau, Explicit fermionic cluster expansion, *Lett. Math. Phys.* **44**:77–88 (1998).
16. A. Abdesselam and V. Rivasseau, Trees, forests and jungles: a botanical garden for cluster expansions, in *Constructive Physics*, V. Rivasseau, ed., Lecture Notes in Physics, Vol. 446 (Springer Verlag, 1995).
17. V. Rivasseau, *From Perturbative to Constructive Renormalization* (Princeton University Press, 1991).
18. M. Disertori and V. Rivasseau, Continuous constructive fermionic renormalization, *Ann. H. Poincaré* **1**:1 (2000).
19. M. Gevrey, Sur la nature analytique des solutions des équations aux dérivées partielles, (Ann. Scient. Ec. Norm. Sup., 3 série. t. 35, p. 129–190), in *Oeuvres de Maurice Gevrey*, pp. 243 (CNRS, 1970).
20. A. Lesniewski, Effective action for the Yukawa_2 quantum field theory, *Commun. Math. Phys.* **108**:437 (1987).
21. C. de Calan and V. Rivasseau, Local existence of the Borel transform in euclidean Φ_4^4, *Commun. Math. Phys.* **82**:69 (1981).